Technical Comments

Comment on "A New Integral Calculation of Skin Friction on a Porous Plate"

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In a recent paper[1] it is claimed that a refined Kármán Pohlhausen (K.P.) method, based on a double integration, in the direction normal to the external mainstream, of the incompressible, zero pressure gradient boundary-layer equation

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{1}$$

leads to accurate and reliable results for the problem of determining the skin friction on a porous plate. The purpose of this Comment is to indicate that the method used is not new, but that it is a standard variation of the K.P. method and indeed is a variation which has already been explored.[2]

From the incompressible boundary-layer equation an ordinary differential equation for the boundary layer thickness $\delta(x)$ may be obtained by multiplying by $y^m u^n$, where m and n are integers, and integrating the resulting equation with respect to y from 0 to δ. By substituting a velocity profile which is defined in terms of δ an ordinary differential equation with δ as dependent variable can be obtained. If the assumed velocity profile contains k unknown functions then k equations can be obtained by taking k possible combinations of m and n. The combination $m = n = 0$ has special physical significance and the K.P. ordinary differential equation can in fact be derived without reference to the boundary-layer equation itself.[3] A comprehensive account of the development of the subject may be found in Ref. 2; the concensus of opinion, evolved with experience over the years, is that such methods will in general prove reliable for flows in which the pressure gradient is favorable but will be unreliable for flows with adverse pressure gradient. More sophisticated integral techniques do not suffer from this disadvantage.[4]

If Eq. (1) be multiplied by y, if the resulting equation be integrated over the boundary layer, and if v be eliminated by means of the equation

$$v(x, y) = v_w(x) - \int_0^y \frac{\partial u}{\partial x}\,dy$$

then we obtain, after simple integration by parts, the equation

$$\int_0^\delta y\frac{\partial}{\partial x}u^2 dy + \int_0^\delta dy\, u(x, y)\int_0^y \frac{\partial}{\partial x}u(x, \theta)\, d\theta = \delta u_0 \int_0^\delta \frac{\partial u}{\partial x}dy - u_0\nu - v_w\left[\delta u_0 - \int_0^\delta u dy\right]$$

This is the equation to which Eq. (2) of Ref. reduces if the first term of that equation is integrated by parts.

To see why the author's results are superior to the standard K.P. results we need only look at the form of Eq. (1) at the wall, i.e.

$$v_w \left.\frac{\partial u}{\partial y}\right|_{y=0} = \nu \left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0} \tag{2}$$

Here, a significant error is introduced into the K.P. solution since Eq. (2) is not satisfied by either of the profiles for u/u_0 selected in Ref. 1. If, on the other hand, the boundary-layer equation be multiplied by y prior to integration over the boundary layer, the error introduced near the wall will be diminished since the equation is now satisfied at the wall.

Table 1, compares, for the case of zero mass transfer, the results for τ_0, the wall shear stress, obtained with various profiles $f(\eta)$ and for various combinations of m and n.

Table 1 Comparison of skin-friction results for various K.P. formulations (case of zero mass transfer)

m	n	$f(\eta)$	$\frac{\tau_0}{\rho u_0^2}\left(\frac{u_0 x}{\nu}\right)^{1/2}$	Ref.
0	0	η	0.289	3
0	1	η	0.250[a]	1
0	0	$2\eta - 2\eta^3 + \eta^4$	0.343	3
0	1	$2\eta - 2\eta^3 + \eta^4$	0.350[a]	1
0 1	0 0	$1 - (1 - \eta)^3(1 + \lambda\eta)$	0.3314	5
		Exact solution	0.3321	

[a] As calculated from $\tau_0 = (\mu/\delta)u_0 f'(0)$.

References

[1] Zien, T. F., "A New Integral Calculation of Skin Friction on a Porous Plate," *AIAA Journal*, Vol. 9, No. 7, July 1971, pp. 1423–1425.

[2] Rosenhead, L. R., ed., *Laminar Boundary Layers*, Oxford University Press, Oxford, England, 1963, pp. 317–318 and Chapt. VI.

[3] Goldstein, S., ed., *Modern Developments in Fluid Dynamics*, Vol. 1, Dover, New York, 1965, pp. 131–132 and 157.

[4] MacDonald, D. A., "Solution of the Incompressible Boundary Layer Equations via the Galerkin Kantorovich Technique," *J. Inst. Maths. Applics.*, Vol. 6, 1970, pp. 115–130.

[5] Sutton, W. G. L., "An approximate Solution of the Boundary Layer Equations for a Flat Plate," *The Philosophical Magazine*, Vol. 23, Ser. 7, pp. 1146–1152.

Received March 31, 1972.
Index category: Boundary Layers and Convective Heat Transfer—Laminar.
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Reply by Author to D. A. MacDonald

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The statement by MacDonald that the method used in Ref. 1 is not new and it is one of the previously explored variations of the K-P method is both inaccurate and misleading. The earlier brief reply to Granville by the author[2] and a subsequent paper[3] by the author together with a careful digest of Ref. 1 actually suffice to clarify the issue. However, for the benefit of other casual readers, it is felt that the following remarks are perhaps worthwhile.

First of all, this author apparently has failed to make clear the central idea of the present method and has thus regrettably

Received July 11, 1972.
Index category: Boundary Layers and Convective Heat Transfer–Laminar.
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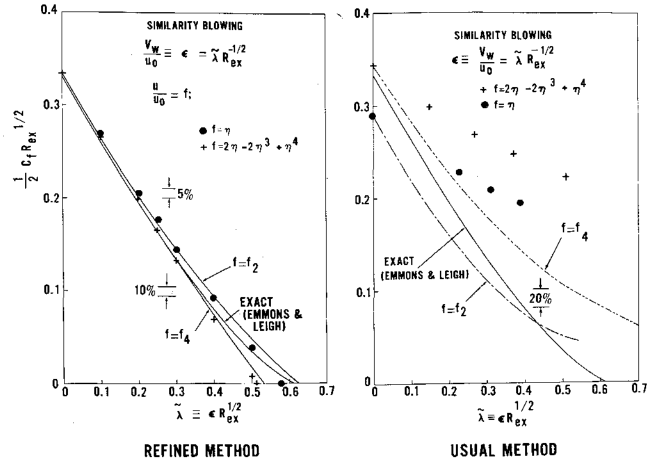

Fig. 1 Skin friction on a porous plate with similarity blowing; f_2 and f_4 are polynomial profiles of 2nd and 4th degrees, respectively, satisfying the compatibility condition (see Zien[3]).

misled MacDonald. This fact is evidenced by the footnote of MacDonald's Table 1, which states clearly that those skin-friction results in the Table, incorrectly quoted as those of Ref. 1†, are obtained by taking the local slope of the assumed velocity profile at the wall. These improper results are apparently based solely on the single equation, i.e., the double-integration equation. It is emphasized here that Volkov's original method,[4] of which Ref. 1 is a direct generalization to allow for surface mass transfer, is to calculate the skin friction by considering the momentum balance across the entire boundary layer, i.e., Eq. (3) of Ref. 1, instead of taking the local slope of the assumed profile at the wall. Of course, the two methods of calculation would lead to the same exact answer if the exact profile were used. It is in this respect that the present method differs from other methods which also exploit the idea of the double integration. The present method is basically a one-parameter type of integral method. However, use is made of two equations, i.e., Eqs. (2) and (3) of Ref. 1, generated by using both the first and the second integration of the original (differential) momentum equation. Only one ordinary differential equation results from this approach, i.e., Eq. (2), which determines δ; C_f then follows immediately from an algebraic equation, Eq. (3). In addition to the standard text books cited by the commentator, the survey paper by Libby et al.[5] is particularly pertinent to the present discussion. The alternate derivation of the double-integration equation given in the Comment is also indicated in this paper.

Secondly, MacDonald's assertion that the improvement of skin-friction calculation due to the present refinement is connected with the satisfaction of the compatibility condition in some modified fashion is of questionable validity. The information contained in Ref. 3, which was also mentioned in a footnote of Ref. 1, indicates plainly that, with the same profiles all satisfying the compatibility condition, the original K-P method still yields results of much poorer quality than the present method. Some typical results are shown in Fig. 1 here which is taken from Zien.[6] The reliability as well as the accuracy of the present method is thus further demonstrated. Mathematically speaking, the imposition of the compatibility condition as a boundary condition to the original boundary-layer equations is obviously unsound. As a matter of fact, previous investigators of integral methods, notably Tani,[7] have shown that the abandonment of such a condition in choosing a profile could sometimes lead to better results.

Thirdly, in regard to the question of the reliability of the present method in treating flows with adverse pressure gradients, the results reported in Ref. 3 should provide a plausible, if not conclusive, answer.

In conclusion, it is the author's opinion that Volkov's procedure combines simplicity and accuracy to an unusual degree, and therefore is worthy of further exploration and extension. It is regretted that MacDonald's comment evidently results from a combination of misinterpretations and misquotations of Volkov's[4] and the author's[1] work.

† The numbers, if properly obtained through the present method, should be 0.333 and 0.336 for $f = \eta$ and $f = 2\eta - 2\eta^3 + \eta^4$, respectively, see Volkov[4] or Zien.[1] Note also that apparently the numbers of m and n associated with "Ref. 1" in the Table are, by mistake, interchanged.

References

[1] Zien, T. F., "A New Integral Calculation of Skin Friction on a Porous Plate," *AIAA Journal*, Vol. 9, No. 7, July 1971, pp. 1423–1425.

[2] Zien, T. F., "Reply by Author to P. S. Granville," *AIAA Journal*, Vol. 10, No. 3, March 1972, p. 367.

[3] Zien, T. F., "Skin Friction on Porous Surfaces Calculated by a Simple Integral Method," *AIAA Journal*, Vol. 10, No. 10, Oct. 1972, pp. 1267–1268.

[4] Volkov, V. N., "A Refinement of the Karman-Pohlhausen Integral Method in Boundary-Layer Theory," *Journal of Engineering Physics*, Vol. 9, No. 5, 1965, pp. 371–374.

[5] Libby, P. A., Morduchow, M. and Bloom, M., "Critical Study of Integral Methods in Compressible Laminar Boundary-Layers," TN 2655, March 1952, NACA.

[6] Zien, T. F., "Study of Boundary-Layer Flows with Pressure Gradient and Mass Transfer by a Simple Integral Method," NOLTR 71–51, March 1971, Naval Ordnance Lab., Silver Spring, Md.

[7] Tani, I., "On the Approximate Solution of the Laminar Boundary-Layer Equations," *Journal of the Aeronautical Sciences*, Vol. 21, July 1954, pp. 487–495 and 504.

Comment on "Calculation of Three-Dimensional Laminar Boundary-Layer Flows"

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IN their recent paper,[1] Fillo and Burbank presented results of computation of three-dimensional, laminar boundary layer for flow past a flat plate with attached cylinder. Comparisons were made between results of exact calculation and that based on the approximating scheme introduced by Wang.[2] It was concluded that good agreement exists in shear stresses and in the prediction of flow reversal for certain regions. However, the approximation as applied failed to predict the flow reversal for flows at a distance of 7.32 cm or greater from the line of symmetry.

It may be pointed out here that Wang's approximating scheme will yield better predictions if a streamline coordinate system (based on the inviscid streamline at the edge of the boundary layer) is used instead of Cartesian coordinates. As discussed in their paper, the term $w u_z$† is important to the flow reversal. Its value, however, differs considerably depending on which coordinate system is used. In Fig. 1, velocities and their components in both Cartesian and streamline coordinates (denoted by subscript s) are shown. Clearly, $u_s \approx u$, $w_s = 0$ for the inviscid flow and $u_s \approx u$, $w_s \ll w$ inside the boundary layer. One can expect, then, that the term $(w u_z)_s$ will also be much smaller than $w u_z$.

Received April 26, 1972.
Index category: Boundary Layers and Convective Heat Transfer—Laminar.

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† Symbols follow those of Ref. 1 except where noted.